# Chapter 1

# Rovnice se separovanymi promennymi

Postupovat budeme presne podle kroku uvedenych na prednasce. Ke krokum uvedenym na prednasce (kroky 1 az 6) pridame v nekterych pripadech jeste nasledujici kroky:

(0) Prevedeni puvodni rovnice na rovnici se separovanymi promennymi a urceni funkci $g$ a $h$.

(7) V pripade, ze puvodni rovnice nebyla rovnice se separovanymi promennymi a museli jsme pouzit krok (0), tak lepe vysetrit lepeni z kroku (6). Nektera reseni, ktera se protinaji se nemuseji dat slepit. Nebo-li, zjistit nejenom to, zda se v lepicich bodech rovnaji limity funkcnich hodnot, ale take, zda se tam rovnaji limity hodnot derivace (viz priklad 7).

(8) V pripade dodatecnych podminek dopocitat, ktera reseni splnuji tyto podminky (napr. pocatecni podminku).

V pisemce je krome vyse popsaneho postupu tez potreba nacrtnout obrazek toho, jak vypadaji maximalni reseni.

**Poznamka 1.1** (Symetrie)**.** *Necht $y(x)$, $x \in I$ je maximalnim resenim rovnice $y' = g(y)h(x)$ a $T \in \mathbb{R}$.*

*(a) Jestlize je $h(x)$ licha, pak $H(x)$ je suda a funkce $\tilde{y}(x) = y(-x)$, $x \in -I$ je taktez maximalni reseni.*

*(b) Jestlize je $g(y)$ licha, pak $G(y)$ je suda a funkce $\tilde{y}(x) = -y(x)$, $x \in I$ je taktez maximalni reseni.*

*(c) Jestlize jsou $h(x)$ a $g(y)$ sude, pak $H(x)$ a $G(y)$ jsou liche a funkce $\tilde{y}(x) = -y(-x)$, $x \in -I$ je taktez maximalni reseni.*

*(d) Jestlize je $h(x)$ $T$-periodicka, pak $H(x)$ je take $T$-periodicka a pro kazde $k \in \mathbb{Z}$ je funkce $y_k(x) = y(x + kT)$, $x \in I - kT$ taktez maximalni reseni.*

*(e) Jestlize je $g(y)$ $T$-periodicka, pak $G(y)$ je take $T$-periodicka a pro kazde $k \in \mathbb{Z}$ je funkce $y_k(x) = y(x) + kT$, $x \in I$ taktez maximalni reseni.*

### 1.0.1 $y' = y$, $y(0) = 1$

(0) $g(y) = y$, $h(x) = 1$.

(1) $I = \mathbb{R}$.

(2) $g(y) = 0$ prave tehdy kdyz $y = 0$, $y_{st}(x) = 0$, $x \in \mathbb{R}$.

(3) $J^+ = \mathbb{R}^+$, $J^- = \mathbb{R}^-$.

(4) $H(x) = x$, $G(y) = \log|y|$.

(5) $M_c^{\pm} = \{x \in I;\ H(x) + c \in G(J^{\pm}\} = \mathbb{R}$, $y^{\pm}(x) = G^{-1}(H(x) + c) = \pm e^{x+c}$. Obecne reseni ma tedy tvar $y_K(x) = Ke^x$, $K \in \mathbb{R}$ ($K = \pm e^c$), $x \in \mathbb{R}$.

(6) Definicni obory reseni jsou $\mathbb{R}$, tedy nelze lepit a reseni z (5) jsou maximalni reseni.

(8) $y(x) = e^x$, $x \in \mathbb{R}$.

### 1.0.2 $yy' + xy^2 = x$, $y(1) = 0$

(0) Lze resit bud primo prevedenim na rovnici $y' = \frac{x-xy^2}{y}$ nebo substituci $y^2 = z$, coz vede na rovnici $z' = 2x(1 - z)$. Budeme resit primo, tedy $g(y) = \frac{1-y^2}{2y}$, $h(x) = 2x$.

(1) $I = \mathbb{R}$.

(2) $g(y) = 0$ prave tehdy kdyz $y = \pm 1$, $y_{st}^{\pm}(x) = \pm 1$, $x \in \mathbb{R}$.

(3) $J^1 = (-\infty, -1)$, $J^2 = (-1, 0)$, $J^3 = (0, 1)$, $J^4 = (1, +\infty)$.

(4) $H(x) = x^2$, $G(y) = -\log|1 - y^2|$.

(5) $M_c^i = \left\{x \in I;\ H(x) + c \in G\left(J^i\right)\right\}$. Tedy

$$\begin{aligned} M_c^1 = M_c^4 = &\quad \mathbb{R}, & c \in \mathbb{R},\\ M_c^{2,-} = M_c^{3,-} = &\quad (-\infty, -\sqrt{-c}), & c \leq 0,\\ M_c^{2,+} = M_c^{3,+} = &\quad (\sqrt{-c}, +\infty), & c \leq 0,\\ M_c^2 = M_c^3 = &\quad \mathbb{R}, & c > 0. \end{aligned}$$

$y_c^{\pm}(x) = G^{-1}(H(x) + c)$. Tedy

$$\begin{aligned} y_c^1(x) = &\ -\sqrt{1 + e^{-x^2-c}}, & c \in \mathbb{R},\ x \in \mathbb{R},\\ y_c^4(x) = &\ \sqrt{1 + e^{-x^2-c}}, & c \in \mathbb{R},\ x \in \mathbb{R}\\ y_c^{2,\pm}(x) = &\ -\sqrt{1 - e^{-x^2-c}}, & c \leq 0,\ x \in M_c^{2,\pm},\\ y_c^{3,\pm}(x) = &\ \sqrt{1 - e^{-x^2-c}}, & c \leq 0,\ x \in M_c^{3,\pm},\\ y_c^2(x) = &\ -\sqrt{1 - e^{-x^2-c}}, & c > 0,\ x \in \mathbb{R},\\ y_c^3(x) = &\ \sqrt{1 - e^{-x^2-c}}, & c > 0,\ x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

(6) Vzhledem ke tvaru definicnich oboru jsou jedini kandidati na lepeni $y_0^{2,\pm}$ a $y_0^{3,\pm}$.

(7) Po dopocitani limit derivaci funkci $y_0^{i,\pm}$ v bode 0 dostavame, ze se daji slepit jen ty s opacnym znamenkem. Tedy dostavame maximalni reseni

$$y_0^{\pm}(x) = \pm\,\mathrm{sign}(x)\sqrt{1 - e^{-x^2-c}},\ x \in \mathbb{R}.$$

(8) Dosazenim do rovnice ze zadani okamzite dostavame, ze reseni splnujici pocatecni podminku neexistuje.

### 1.0.3 $y' = \frac{y^2}{x^2}$, $y$ je maximalni reseni, ktere je omezene na svem definicnim oboru

(0) $g(y) = -y^2$, $h(x) = -\frac{1}{x^2}$.

(1) $I_1 = \mathbb{R}^+$, $I_2 = \mathbb{R}^-$.

(2) $g(y) = 0$ prave tehdy kdyz $y = 0$, $y^i_{st}(x) = 0$, $x \in I_i$, $i = 1, 2$.

(3) $J^1 = \mathbb{R}^+$, $J^2 = \mathbb{R}^-$.

(4) $H(x) = \frac{1}{x}$, $G(y) = \frac{1}{y}$.

(5) $M^{i,j}_c = \left\{x \in I_i;\ H(x) + c \in G\left(J^j\right)\right\}$. Tedy

$$M^{i,j}_c = \begin{cases} I_1 : & i = j = 1,\ c \geq 0 \\ \left(0, -\frac{1}{c}\right) : & i = j = 1,\ c < 0, \\ \emptyset : & i = 1, j = 2,\ c \geq 0, \\ \left(-\frac{1}{c}, +\infty\right) : & i = 1, j = 2,\ c < 0, \\ \left(-\infty, -\frac{1}{c}\right) : & i = 2, j = 1,\ c > 0, \\ \emptyset : & i = 2, j = 1,\ c \leq 0, \\ \left(-\frac{1}{c}, 0\right) : & i = 2, j = 2,\ c > 0, \\ I_2 : & i = 2, j = 2,\ c \leq 0. \end{cases}$$

Obecne reseni ma tedy tvar $y^{i,j}_c(x) = G^{-1}(H(x) + c) = \frac{x}{cx+1}$, $x \in M^{i,j}_c$.

(6) Reseni evidentne nelze lepit ani v bode 0, kde neni definovana puvodni rovnice, ani v bode $-\frac{1}{c}$, kde reseni nemaji vlastni limitu.

(8) $y^i_{st}(x)$, $y^{1,1}_c$, pro $c > 0$ a $y^{2,2}_c$ pro $c < 0$.

### 1.0.4 $y' = -\frac{(1+y^2)x}{1+x^2}$, $y(0) = 1$

(0) $g(y) = 1 + y^2$, $h(x) = -\frac{x}{1+x^2}$.

(1) $I = \mathbb{R}$.

(2) $g(y) \neq 0$, $y \in \mathbb{R}$.

(3) $J = \mathbb{R}$.

(4) $H(x) = -\frac{1}{2}\log(1 + x^2)$, $G(y) = \arctan(y)$.

(5) $M_c = \left\{x \in I;\ H(x) - \frac{c}{2} \in G(J)\right\}$. $G(J) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, tedy

$$\begin{aligned} M_c &= & \emptyset & \quad : c \geq \pi, \\ M_c &= & \left(-\sqrt{e^{\pi-c} - 1}, \sqrt{e^{\pi-c} - 1}\right) & \quad : c \in (-\pi, \pi), \\ M^-_c &= & \left(-\sqrt{e^{\pi-c} - 1}, -\sqrt{e^{-\pi-c} - 1}\right) & \quad : c \leq -\pi, \\ M^+_c &= & \left(\sqrt{e^{-\pi-c} - 1}, \sqrt{e^{\pi-c} - 1}\right) & \quad : c \leq -\pi. \end{aligned}$$

$y_c(x) = G^{-1}\left(H(x) - \frac{c}{2}\right)$. Obecne reseni ma tedy tvar $y_c(x) = \tan\left(-\frac{1}{2}\left(c + \log\left(1 + x^2\right)\right)\right)$, $c \in (-\pi, \pi)$, $x \in M_c$, $y^{\pm}_c(x) = \tan\left(-\frac{1}{2}\left(c + \log\left(1 + x^2\right)\right)\right)$, $c \leq -\pi$, $x \in M^{\pm}_c$.

(6) Jedini kandidati na lepeni jsou $y^+_{-\pi}$ a $y^-_{-\pi}$ v bode 0. V tomto bode vsak maji nevlastni limitu, tedy je nelze slepit.

(8) Pocatecni podminku splnuje $y_{-\frac{\pi}{2}}(x)$.

### 1.0.5 $y' = \sqrt[5]{y^2}$, $y(-14) = -32$, $y(0) = 1$

(0) $g(y) = \sqrt[5]{y^2}$, $h(x) = 1$.

(1) $I = \mathbb{R}$.

(2) $g(y) = 0$ prave tehdy kdyz $y = 0$, $y_{st}(x) = 0$, $x \in \mathbb{R}$.

(3) $J^+ = \mathbb{R}^+$, $J^- = \mathbb{R}^-$.

(4) $H(x) = x$, $G(y) = \frac{5}{3} y^{\frac{3}{5}}$.

(5) $M_c^{\pm} = \{x \in I;\ H(x) + c \in G\left(J^{\pm}\right)\}$. Tedy

$$\begin{aligned} M_c^+ = &\quad (-c, +\infty) &&: c \in \mathbb{R}, \\ M_c^- = &\quad (-\infty, -c) &&: c \in \mathbb{R}. \end{aligned}$$

Obecne reseni ma tedy tvar $y_c^{\pm}(x) = G^{-1}(H(x) + c) = \left(\frac{3}{5}(x + c)\right)^{\frac{5}{3}}$, $x \in M_c^{\pm}$.

(6) $\lim_{x\to -c} y_c^{\pm}(x) = \lim_{x\to -c} y_{st}(x) = 0$. Tedy se daji kladna a zaporna reseni navzajem lepit a take se daji lepit spolu se stacionarnim resenim. Obecne maximalni reseni tedy maji tvar:

$$\begin{aligned}
y_{st}(x) &= \quad 0 && : x \in \mathbb{R}, \\
\widetilde{y}_c(x) &= \begin{cases} \left(\frac{3}{5}(x + c)\right)^{\frac{5}{3}} & : x < -c, \\ 0 & : x \geq -c, \end{cases} && : c \in \mathbb{R} \\
\widehat{y}_d(x) &= \begin{cases} 0 & : x \leq -d, \\ \left(\frac{3}{5}(x + d)\right)^{\frac{5}{3}} & : x > -d, \end{cases} && : d \in \mathbb{R} \\
y_{c,d}(x) &= \begin{cases} \left(\frac{3}{5}(x + c)\right)^{\frac{5}{3}} & : x < -c, \\ 0 & : x \in [-c, -d], \\ \left(\frac{3}{5}(x + d)\right)^{\frac{5}{3}} & : x > -d, \end{cases} && : c, d \in \mathbb{R},\ c \geq d.
\end{aligned}$$

(8) Z podminek plyne, ze reseni neni ani nezaporne ani nekladne a tedy se jedna o reseni typu $y_{c,d}(x)$. Z podminky $y(-14) = -32$ plyne $c = \frac{2}{3}$ a z podminky $y(0) = 1$ plyne $d = \frac{5}{3}$. Tedy $c < d$ a takove reseni neexistuje.

### 1.0.6 $y' = \frac{\cos(x)}{e^y}$

(0) $g(y) = e^{-y}$, $h(x) = \cos(x)$.

(1) $I = \mathbb{R}$.

(2) $g(y) \neq 0$, $y \in \mathbb{R}$.

(3) $J = \mathbb{R}$.

(4) $H(x) = \sin(x)$, $G(y) = e^y$.

(5) $M_c = \{x \in I;\ H(x) + c \in G(J)\}$. $G(J) = \mathbb{R}^+$, tedy

$$\begin{aligned} M_c = &\quad \emptyset &&: c \leq -1,\\ M_c = &\quad \mathbb{R} &&: c > 1,\\ M_c^k = &\quad (2k\pi - \arcsin(c), (2k+1)\pi + \arcsin(c)) &&: c \in (-1, 1],\ k \in \mathbb{Z}. \end{aligned}$$

$y_c(x) = G^{-1}(H(x) + c)$. Obecne reseni ma tedy tvar

$$\begin{aligned} y_c(x) = &\quad \log(\sin(x) + c) &&: x \in \mathbb{R},\ c > 1,\\ y_c^k(x) = &\quad \log(\sin(x) + c) &&: x \in M_c^k,\ c \in (-1, 1]. \end{aligned}$$

(6) Reseni $y_c^k$ maji na kraji svych definicnich oboru nevlastni limity, tedy nelze lepit.

### 1.0.7 $y'(2 - e^x) = -3e^x \tan(y)\cos^2(y)$. Pro ktera $A \in \mathbb{R}$ existuje reseni splnujici: $\lim_{x\to\infty} y(x) - \lim_{x\to-\infty} y(x) = A$?

(0) Prevedeme rovnici na rovnici:

$$y'(x) = -\frac{3e^x \tan(y)\cos^2(y)}{2 - e^x}, \tag{1.2}$$

pak $g(y) = \tan(y)\cos^2(y)$, $h(x) = -\frac{3e^x}{2-e^x}$. Funkce $g$ je $\pi$-periodicka, tedy se omezime pouze na vysetrovani $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ a na zaver vyuzijeme periodicitu k nalezeni vsech reseni.

(1) $I_1 = (-\infty, \log(2))$, $I_2 = (\log(2), +\infty)$.

(2) $g(y) = 0$ prave tehdy kdyz $y = 0$, $y_{st}^i(x) = 0$, $x \in I_i$, $i = 1, 2$.

(3) $J^1 = \left(-\frac{\pi}{2}, 0\right)$, $J^2 = \left(0, \frac{\pi}{2}\right)$.

(4) $H(x) = 3\log|2 - e^x|$, $G(y) = \log|\tan(y)|$.

(5) $M_c^{i,j} = \left\{x \in I_i;\ H(x) + c \in G(J^j)\right\} = I_i$, $y_K^{i,j}(x) = G^{-1}(H(x)+c) = \arctan\left(K(2 - e^x)^3\right)$, $x \in I_i$, $K \in (0, +\infty)$ pro $i \neq j$ a $K \in (-\infty, 0)$ pro $i = j$, nebot $K = \pm e^c$. Obecne reseni ma tedy tvar $y_K^i(x) = \arctan\left(K(2 - e^x)^3\right)$, $K \in \mathbb{R}$, $x \in I_i$.

(6) V prevedene rovnici (1.2) se lepit neda, neb jsou reseni definovana na maximalnich podintervalech definicniho oboru teto rovnice.

(7) V puvodni rovnici lze vsechny reseni navzajem slepit v bode $\log(2)$, nebot $\lim_{x\to\log(2)} y_K^i(x) = \lim_{x\to\log(2)} (y_K^i)'(x) = 0$, kde limity jsou zleva pro $i = 1$ a zprava pro $i = 2$. Pokud vezmeme v potaz na zacatku zminenou $\pi$-periodicitu funkce $g$, pak obecne maximalni reseni ma tvar :

$$y_{K,T,k}(x) = \begin{cases} \arctan\left(K(2 - e^x)^3\right) + k\pi & : x \leq \log(2),\ K \in \mathbb{R},\ k \in \mathbb{Z}\\ \arctan\left(T(2 - e^x)^3\right) + k\pi & : x > \log(2),\ T \in \mathbb{R},\ k \in \mathbb{Z}. \end{cases}$$

(8) Podminka je splnena pro $A \in (-\pi, \pi)$.

### 1.0.8 $\quad yy' = \frac{1-2x}{y}$

(0) $g(y) = \frac{1}{y^2}$, $h(x) = 1 - 2x$.

(1) $I = \mathbb{R}$.

(2) $g(y) \neq 0$, $y \in \mathbb{R}$.

(3) $J^+ = \mathbb{R}^+$, $J^- = \mathbb{R}^-$.

(4) $H(x) = x - x^2$, $G(y) = \frac{1}{3}y^3$.

(5) $M_c^\pm = \{x \in I;\ H(x) + c \in G(J^\pm\}$. Tedy

$$
\begin{aligned}
M_c^+ &= \begin{cases} \emptyset & : c \leq -\frac{1}{4}, \\ \left(\frac{1-\sqrt{1+4c}}{2}, \frac{1+\sqrt{1+4c}}{2}\right) & : c > -\frac{1}{4}, \end{cases} \\
M_c^- &= \qquad \mathbb{R} \qquad : c < -\frac{1}{4}, \\
M_c^{-,1} &= \left(-\infty, \frac{1-\sqrt{1+4c}}{2}\right) \qquad : c \geq -\frac{1}{4}, \\
M_c^{-,2} &= \left(\frac{1+\sqrt{1+4c}}{2}, +\infty\right) \qquad : c \geq -\frac{1}{4}.
\end{aligned}
$$

$y_c^\pm(x) = G^{-1}(H(x)+c)$. Obecna reseni maji tedy tvar $y_c^\pm(x) = \sqrt[3]{-3x^2 + 3x + 3c}$, $x \in M_c^\pm$, $y_c^{-,i}(x) = \sqrt[3]{-3x^2 + 3x + 3c}$, $x \in M_c^{-,i}$ pro prislusne indexy $+, -, c$ a $i$.

(6) Reseni maji na kraji svych definicnich oboru nevlastni limity a tedy je nelze lepit.

### 1.0.9 $\quad xy' + y = y^2$

(0) Tato rovnice se da resit primo prevedenim na rovnici $y' = \frac{y^2-y}{x}$ a nebo substituci $z = xy$. My pouzijeme substituci. Pomoci teto substituce prevedeme puvodni rovnici na novou rovnici:

$$z' = \frac{z^2}{x^2}.$$

Tuto rovnici jsme jiz vyresili v 3. prikladu. Pouzijeme tedy toto reseni a dopocitame $y = \frac{z}{x}$.

(7) Reseni z kroku (0) lze slepit v 0. Tedy dostavame tyto reseni:

$$
\begin{aligned}
y_{st}^i(x) &= \quad i \qquad : i = 0, 1,\ x \in \mathbb{R} \\
y_{c,i}(x) &= \frac{1}{cx+1} \quad c \in \mathbb{R} \setminus \{0\},\ x \in \begin{cases} \left(-\infty, -\frac{1}{c}\right) & : i = 1, \\ \left(-\frac{1}{c}, +\infty\right) & : i = 2. \end{cases}
\end{aligned}
$$

### 1.0.10 $\quad y' = 10^{x+y}$

(0) $g(y) = 10^y$, $h(x) = 10^x$.

(1) $I = \mathbb{R}$.

(2) $g(y) \neq 0$, $y \in \mathbb{R}$.

(3) $J = \mathbb{R}$.

(4) $H(x) = \frac{10^x}{\log(10)}$, $G(y) = -\frac{1}{10^y \log(10)}$.

(5) $M_c = \Big\{x \in I;\ H(x) - \frac{c}{\log(10)} \in G(J)\Big\}$. Tedy

$$M_c = \begin{cases} \emptyset & : c \leq 0, \\ (-\infty, \log_{10}(c)) & : c > 0. \end{cases}$$

$y_c(x) = G^{-1}\Big(H(x) - \frac{c}{\log(10)}\Big)$. Obecne reseni ma tedy tvar $y_c(x) = -\log_{10}(c - 10^x)$, $c > 0$, $x \in M_c$.

(6) Reseni maji na kraji svych definicnich oboru nevlastni limity a tedy je nelze lepit.

### 1.0.11 $e^{-y}(1 + y') = 1$

(0) Prevedeme na ekvivalentni rovnici

$$y' = e^y - 1.$$

Tedy, $g(y) = e^y - 1$, $h(x) = 1$.

(1) $I = \mathbb{R}$.

(2) $g(y) = 0$ prave tehdy kdyz $y = 0$, $y_{st}(x) = 0$, $x \in \mathbb{R}$.

(3) $J^+ = \mathbb{R}^+$, $J^- = \mathbb{R}^-$.

(4) $H(x) = x$, $G(y) = \log|e^y - 1| - y$.

(5) $M_c^\pm = \{x \in I;\ H(x) + c \in G(J^\pm)\} = \mathbb{R}$, $G(J^-) = \mathbb{R}$ a $G(J^+) = \mathbb{R}^-$. Tedy

$$\begin{aligned} M_c^- = &\quad \mathbb{R} && : c \in \mathbb{R}, \\ M_c^+ = &\quad (-\infty, -c) && : c \in \mathbb{R}. \end{aligned}$$

Tedy $y_c^\pm(x) = G^{-1}(H(x) + c) = -\log(1 \mp e^{x+c})$, $c \in \mathbb{R}$, $x \in M_c^\pm$. Polozime $K = \mp e^c$. Obecne reseni ma tedy tvar

$$y_K(x) = \quad -\log(1 + Ke^x) \begin{cases} x \in \mathbb{R} & : K \geq 0, \\ x \in (-\infty, -\log(-K)) & : K < 0. \end{cases}$$

(6) Reseni maji na kraji svych definicnich oboru nevlastni limity a tedy je nelze lepit.

### 1.0.12 $y' \sin(x) = g(y)$, kde $g(y) = y \log(y)$ pro $y > 0$ a $g(0) = 0$

(0) Prevedeme puvodni rovnici na novou rovnici

$$y' = \frac{g(y)}{\sin(x)}, \tag{1.3}$$

tedy $h(x) = \frac{1}{\sin(x)}$.

(1) Funkce $h(x)$ je $2\pi$ periodicka, tedy budeme resit jen pro $x \in (-\pi, \pi)$ a na zaver vyuzijeme periodicitu k nalezeni vsech reseni. $I_1 = (-\pi, 0)$, $I_2 = (0, \pi)$.

(2) $g(y) = 0$ prave tehdy kdyz $y \in \{0, 1\}$. Tedy $y_{st,v}^i(x) = v$, $x \in I_i$, $i \in \{1, 2\}$, $v \in \{0, 1\}$.

(3) $J^1 = (0,1)$, $J^2 = (1, +\infty)$.

(4) $H(x) = \log\left|\tan\left(\frac{x}{2}\right)\right|$, $G(y) = \log|\log(y)|$.

(5) $M_c^{i,j} = \left\{x \in I_i;\ H(x) + c \in G(J^j)\right\} = I_i$, nebot $G(J^j) = \mathbb{R}$, $j = 1, 2$. Obecne reseni ma tedy tvar $y_K^{i,j}(x) = G^{-1}(H(x) + c) = e^{K\tan\left(\frac{x}{2}\right)}$, $x \in I_i$, kde

$$K = \begin{cases} e^c & : i = j, \\ -e^c & : i \neq j, \\ 0 & \text{stacionarni reseni.} \end{cases}$$

Muzeme tedy psat $y_{K,k}^i(x) = e^{K\tan\left(\frac{x}{2}\right)}$, $x \in I_i + 2k\pi$, $K \in \mathbb{R}$, $k \in \mathbb{Z}$.

(6) Reseni rovnice (1.3) nelze lepit, neb v krajnich bodech definicnich oboru neni tato rovnice definovana.

- $(7_1)$ Lepeni v $2k\pi$: $\lim_{x\to 2k\pi} y_{K,k}^i(x) = 1$ pro $K \in \mathbb{R}$, $k \in \mathbb{Z}$, kde pro $i = 1$ se jednalo o limitu zleva a pro $i = 2$ o limitu zprava. $\lim_{x\to 2k\pi}\left(y_{K,k}^i\right)'(x) = \frac{K}{2}$ pro $K \in \mathbb{R}$, $k \in \mathbb{Z}$, kde pro $i = 1$ se jednalo o limitu zleva a pro $i = 2$ o limitu zprava. Tedy spolu muzeme slepit jen reseni se stejnymi $k \in \mathbb{Z}$, $K \in \mathbb{R}$ a dostaneme $y_{K,k}(x) = e^{K\tan\left(\frac{x}{2}\right)}$, $x \in (-\pi, \pi) + 2k\pi$, $K \in \mathbb{R}$, $k \in \mathbb{Z}$.

  V techto bodech lze samozrejme slepit sama se sebou i konstantni 0 (nase druhe stacionarni reseni) a tak i tato reseni mame definovana na $(-\pi,\pi)+2k\pi$, $k \in \mathbb{Z}$.

- $(7_2)$ Lepeni v $(2k+1)\pi$:

$$\begin{aligned}
\lim_{x\to(2k+1)\pi_+} y_{K,k+1}(x) &= & \lim_{x\to(2k+1)\pi_+} \left(y_{K,k}\right)'(x) &= +\infty, & K < 0, k \in \mathbb{Z},\\
\lim_{x\to(2k+1)\pi_+} y_{K,k}(x) &= & \lim_{x\to(2k+1)\pi_+} \left(y_{K,k}\right)'(x) &= 0, & K > 0, k \in \mathbb{Z},\\
\lim_{x\to(2k+1)\pi_+} y_{0,k}(x) &= 1, & \lim_{x\to(2k+1)\pi_+} \left(y_{0,k}\right)'(x) &= 0, & k \in \mathbb{Z},\\
\lim_{x\to(2k+1)\pi_-} y_{K,k}(x) &= & \lim_{x\to(2k+1)\pi_-} \left(y_{K,k}\right)'(x) &= 0, & K < 0, k \in \mathbb{Z},\\
\lim_{x\to(2k+1)\pi_-} y_{K,k}(x) &= & \lim_{x\to(2k+1)\pi_-} \left(y_{K,k}\right)'(x) &= +\infty, & K > 0, k \in \mathbb{Z},\\
\lim_{x\to(2k+1)\pi_-} y_{0,k}(x) &= 1, & \lim_{x\to(2k+1)\pi_-} \left(y_{0,k}\right)'(x) &= 0, & k \in \mathbb{Z}.
\end{aligned}$$

  Lze snadno nahlednout, ze lze slepit obe stacionarni reseni. Rovnez lze slepit zleva reseni se zapornym $K$ a zprava reseni s kladnym $K$. Rovnez lze tato reseni slepit se stacionarnim resenim (konstantni 0). Obecna maximalni reseni maji tedy tvar:

$$\begin{aligned}
y_0(x) &= & 0:\ x \in \mathbb{R},\\
y_1(x) &= & 1:\ x \in \mathbb{R},\\
\widetilde{y}_K^k(x) &= & \begin{cases} 0: & x \leq (2k-1)\pi, \\ e^{K\tan\left(\frac{x}{2}\right)}: & x \in ((2k-1)\pi, (2k+1)\pi), \end{cases} \quad K > 0,\ k \in \mathbb{Z},\\
\widehat{y}_K^k(x) &= & \begin{cases} e^{K\tan\left(\frac{x}{2}\right)}: & x \in ((2k-1)\pi, (2k+1)\pi), \\ 0: & x \geq (2k+1)\pi, \end{cases} ,\ K < 0\ k \in \mathbb{Z},\\
y_{K,T}^{k,l}(x) &= & \begin{cases} e^{K\tan\left(\frac{x}{2}\right)}: & x \in ((2k-1)\pi, (2k+1)\pi), \\ 0: & x \in [(2k+1)\pi, (2l-1)\pi], \\ e^{T\tan\left(\frac{x}{2}\right)}: & x \in ((2l-1)\pi, (2l+1)\pi), \end{cases} \quad K < 0, T > 0, k < l \in \mathbb{Z}.
\end{aligned}$$

### 1.0.13 $y' = \frac{1+y^2}{1+x^2}$, $y(0) = 1$

(0) $g(y) = 1 + y^2$, $h(x) = \frac{1}{1+x^2}$.

(1) $I = \mathbb{R}$.

(2) $g(y) \neq 0$, $y \in \mathbb{R}$.

(3) $J = \mathbb{R}$.

(4) $H(x) = \arctan(x)$, $G(y) = \arctan(y)$.

(5) $M_c = \{x \in I;\ H(x) + c \in G(J)\}$, kde $G(J) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Tedy,

$$M_c = \begin{cases} \emptyset & : c \in \mathbb{R} \setminus (-\pi, \pi), \\ \left(\tan\left(-\frac{\pi}{2} - c\right), +\infty\right) & : c \in (-\pi, 0), \\ \mathbb{R} & : c = 0, \\ \left(-\infty, \tan\left(\frac{\pi}{2} - c\right)\right) & : c \in (0, \pi). \end{cases}$$

Obecne reseni ma tedy tvar $y_c(x) = \tan(\arctan(x) + c)$, $c \in (-\pi, \pi)$, $x \in M_c$.

(6) Reseni maji na kraji svych definicnich oboru nevlastni limity a tedy je nelze lepit a reseni z (5) jsou maximalni reseni.

(8) $y(x) = \tan\left(\arctan(x) + \frac{\pi}{4}\right) = \frac{1+x}{1-x}$, $x \in (-\infty, 1)$.